Measurement of the Surface Inductance and Penetration Depth of Superconducting Aluminum†

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The inductance of a pure-aluminum wire was measured as a function of temperature in the superconducting state at a frequency of 10 MHz. The temperature dependence of the penetration depth calculated from these measurements does not agree with that predicted by the BCS theory for aluminum, an extreme nonlocal superconductor. It is shown that the present measurements agree with previous results at higher frequencies and have a temperature dependence very close to the BCS local theory. It is concluded that there is no rigorous way of obtaining an absolute value of the penetration depth from this or previous measurements of the temperature dependence of the inductance (or the penetration depth).

INTRODUCTION

The penetration depth of bulk aluminum has been the subject of a number of experimental studies.[1-4] Of these, the surface-impedance measurements[1,3,4] were all made at high enough frequencies that the corrections necessitated by phase shifts and the resistive effects of normal electrons were significant. The absorption measurments,[2] also at high frequency, avoided these problems and, in addition, made possible the calculation of the penetration depth without recourse to an assumed temperature dependence. This method did require, however, theoretical extrapolations of the data, which gave rise to some uncertainty in the result. Since aluminum is the prime example of an extreme nonlocal superconductor, it seemed useful to repeat the determination of the penetration depth at a lower frequency. We have used a recently developed technique[5] to measure the temperature-dependent self-inductance of a superconducting aluminum wire at 10 MHz and have attempted to deduce from the results a value of the penetration depth.

THEORY

To make use of the inductance measurements we must relate the self-inductance of a superconducting wire to the penetration depth and relate the change in self-inductance to the measured frequency change of a resonant circuit containing the wire. We can then deduce from the temperature dependence of the measured frequency the temperature dependence of the penetration depth. This quantity, $\lambda(T)$, when compared with purely theoretical expressions, yields a value of $\lambda(0)$, the penetration depth at $T=0$.

The wire is assumed to have a circular cross section, to be straight, and to have a diameter d which is small compared with the length and much greater than the penetration depth. Since a length of about 1.5 m of wire was needed to obtain sufficient sensitivity, the wire was loosely wrapped in a bifilar manner on a quartz supporting frame. However, because the radius of curvature of the bends of the wire was very large compared with the penetration depth and because the regions of curvature were a small fraction of the total length, the assumption of a straight wire does not lead to a significant error. The external inductance (the inductance associated with the energy stored in the magnetic field external to the wire) is assumed to remain constant. Since we are only interested in the temperature dependence of the inductance, this inductance need not be considered. It is true that the azimuthal distribution of the current in the curved portions (and therefore the external inductance) can be a function of temperature, but since the diameter is much greater than the penetration depth, this effect causes only a higher-order correction which can be shown to be negligible.

In the London theory for a local superconductor, the internal magnetic inductance L_M (that associated with the energy stored in the magnetic field within the superconductor) just equals the kinetic inductance L_K (that associated with the kinetic energy of the surface supercurrents), and the total internal inductance is simply

$$L = L_M + L_K = 2L_K = \mu_0 l \lambda_L / \pi d \ . \qquad (1)$$

Here, l is the length and d is the diameter of the wire. The derivation of this result and a discussion of the concept of kinetic inductance has been given previously,[5] where it was also shown that in a superconductor, the rule that total inductance equals magnetic flux/unit current is not always correct. However, for a bulk sample as used in the present measurements, it is shown in the Appendix that the usual analysis yields the correct result. The total inductance per unit surface area is, therefore,

$$L = \mu_0 \lambda \ . \tag{2}$$

This result is derived by using the energy definition of inductance and integrating the Poynting vector at the surface of the superconductor. The result is unaffected by the nonlocal electrodynamics of the superconductor and depends for its validity only on the condition that the thickness or diameter $d \gg \lambda$. When this condition is no longer fulfilled, as for very small wires or thin-film cylinders, the energy definition of total inductance must be used. In such a case the present method of imposing a current along the cylindrical sample measures a different quantity from the usual method of imposing a longitudinal magnetic field. However, for the present experiment, we find $d \gg \lambda$ at all temperatures of interest, and Eq. (2) is valid. Thus for a cylinder of length l and diameter d the inductance is (just as in the local case)

$$L = \mu_0 l \lambda / \pi d \ . \tag{3}$$

As will be discussed in the next section, the losses in the oscillator circuit were small enough that we could simply consider the resonance condition to be

$$\omega_0 = (L_0 C_0)^{-1/2} \ . \tag{4}$$

If it is assumed that the capacitance C_0 is constant and that the change in L with temperature is very small compared with the total inductance L_0 in the circuit, then the differential $d\omega$ is given by

$$d\omega/\omega = -dL/2L_0 \ ,$$

or, using Eq. (3), by

$$d\omega/d\lambda = -\mu_0 l \omega_0 / 2\pi L_0 d \ . \tag{5}$$

Assuming that the penetration depth has a finite value $\lambda(0)$ at $T = 0$, it can quite generally be written in the form

$$\lambda(T) = \lambda(0)\, \theta(T) \ , \tag{6}$$

where θ is a function of the temperature and $\lambda(0) = d\lambda(T)/d\theta$. From Eq. (5), we have

$$\frac{d\omega}{d\theta} = -\frac{\mu_0 l \omega_0}{2\pi L_0 d} \frac{d\lambda}{d\theta}$$

or, solving for $\lambda(0)$,

$$\lambda(0) = \frac{d\lambda}{d\theta} = -\frac{2\pi L_0 d}{\mu_0 l f_0} \frac{df}{d\theta} \ . \tag{7}$$

Here the frequency f has been substituted for the angular frequency, since f is the quantity actually measured.

If the function $\theta(T)$ is the correct temperature dependence of the penetration depth, then $\lambda(0)$ will be a constant as we have assumed. If $\theta(T)$ is not the correct temperature dependence, then we cannot meaningfully identify $d\lambda/d\theta$ as $\lambda(0)$ since it will be a function of T and not a constant. A correct and complete theory of superconductivity would evidently give us both $\lambda(0)$ and $\theta(T)$ in Eq. (6). Before the microscopic theory of superconductivity was developed, it was found empirically that the penetration depth measured in different superconductors approximately fitted the function

$$\lambda(t) = \lambda_y(0)\, y(t) \ , \tag{8}$$

where

$$y(t) = (1 - t^4)^{-1/2} \tag{9}$$

and $t = T/T_c$. When the BCS theory[6] was introduced, a somewhat different function was predicted for the temperature dependence for local superconductors. In this case, for weak-coupling superconductors the theory gives

$$\lambda(t) = \lambda_L(0) Z(t) \ , \tag{10}$$

where

$$Z(t) = \left(\frac{1}{\Delta(t)} \frac{d(\Delta(t)/t)}{d(1/t)} \right)^{-1/2} \ . \tag{11}$$

Here $2\Delta(t)$ is the BCS theory energy gap and $Z(t)$ is a universal function for all superconductors, and has been tabulated by Mühlschlegel.[7]

However, most pure superconductors are nonlocal, and to describe this situation the BCS theory, following Pippard,[8] introduced another parameter, the coherence length, $\xi_0 = \hbar V_F / \pi \Delta(0)$. Here V_F is the Fermi velocity and $2\Delta(0)$ is the energy gap at $T = 0$. In the general case, the expression for the penetration depth is rather complicated. However, in the extreme nonlocal limit, $\xi_0 \gg \lambda$, the penetratior depth can be expressed simply as a universal function of the reduced temperature for all superconductors:

$$\lambda_L{}^*(t) = \lambda_L{}^*(0) Z^*(t) \ , \tag{12}$$

where

$$Z^*(t) = \left[\left(\frac{\Delta(t)}{\Delta(0)} \right) \tanh \left(\frac{3.53 \Delta(t)}{4t\Delta(0)} \right) \right]^{-1/3} \ . \tag{13}$$

Again $2\Delta(t)$ is the BCS energy gap function.

The three temperature dependences mentioned above are shown in Fig. 1. According to the BCS theory, aluminum should qualify as an extreme nonlocal superconductor since $\xi_0/\lambda \approx 30$. Therefore, the temperature dependence of λ should be given very nearly by $Z^*(t)$.[9]

INDUCTANCE MEASUREMENT TECHNIQUE

Tunnel Diode Oscillator

A tunnel diode oscillator operating in a liquid-helium bath furnishes a simple method for measuring the changes in inductance of a superconductor.[5] In addition to its simplicity, the oscillator elimi-

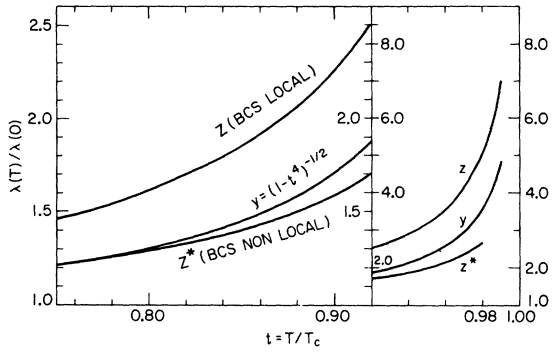

FIG. 1. Temperature dependence of the penetration depth as predicted by the Gorter-Casimir, BCS local, and BCS nonlocal theories. Note the change in vertical scale for reduced temperatures $t > 0.92$.

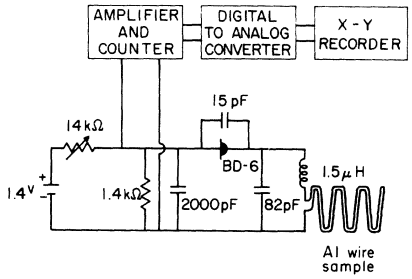

FIG. 2. Tunnel diode oscillator circuit and frequency-measuring electronics.

nates the effect of impedance changes in long leads to the sample and provides the needed sensitivity without excessive power dissipation.

The circuit diagram for the oscillator is shown in Fig. 2. All of the circuit except the mercury battery and the bias resistors is immersed in liquid helium. The bias leads also serve as signal leads. The 10-MHz signal goes to an amplifier and counter and then to a digital-to-analog converter and an x-y recorder. In this experiment, the x axis of the recorder was usually driven with a signal proportional to the vapor pressure of the helium bath. Thus a plot of frequency versus temperature could be obtained. When a long small-diameter superconducting wire was connected in series with the 1.5-μH tank coil of the oscillator, the change of frequency with temperature could be related to the change of internal inductance as a function of the temperature.

The change of frequency can be written

$$\Delta f/f_0 = -(\Delta L/2L_0)(1 - 2/Q^2) - (\Delta R/R)(1/Q^2) , \tag{14}$$

where R is the resistance of the tank circuit and $Q = \omega L/R$. Thus if Q is large enough, the relative change of frequency is given simply by

$$\Delta f/f_0 = -\Delta L/2L_0 . \tag{15}$$

Measurements were made of the Q of a typical oscillator tank circuit by opening the connection between the tunnel diode and the tank circuit and observing the resulting exponential decay of the envelope of the tank circuit voltage. From these measurements, it was determined that if the Q is large enough to allow the oscillator to oscillate, then it is large enough to make the terms in $1/Q^2$ in Eq. (14) negligible. Thus a measurement of Δf gives directly a measurement of ΔL.

The oscillator proved to be stable to about one part in 10^7 and the tank coil inductance was about 2.5×10^{-6} H, so the smallest observable inductance change was about $2L_0\Delta f/f_0 = 5 \times 10^{-13}$ H. For the length of wire used, the change in the inductance over the temperature range of interest was about 10^{-9} H.

Sample Preparation

The samples were 2.5-m-long sections of 99.999% pure aluminum wire, 0.0127-cm diam. To form the sample, the wire was wrapped on a notched quartz plate in a bifilar manner to make the external magnetic inductance as small as possible (Fig. 3). The samples were measured in three conditions: as they were received after drawing, after annealing, and after annealing and chemical polishing. The wire was annealed at 600 °C for various times up to 12 h in a pure-helium atmosphere, obtained from the boil-off of a liquid-helium storage Dewar. The quartz form which supported the loosely held wire was kept in a vertical position both in the furnace and while being mounted into the cryostat, so as to minimize as much as possible

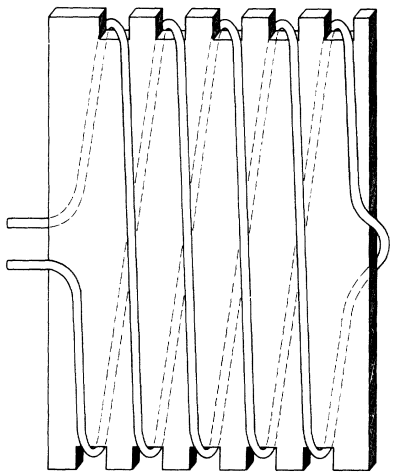

FIG. 3. Schematic drawing of an aluminum wire sample wrapped on a notched quartz plate.

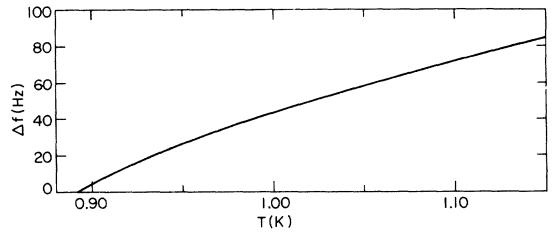

FIG. 4. Temperature dependence of the oscillator frequency with no sample present.

the strain on the annealed wire from its own weight. The wire was chemically polished by immersion for about 1 min in a dilute bath (0.15%) of HF. The annealed samples had a resistance ratio of about 1000 between 300 and 4.2 K, implying the ratio of the mean free path to coherence distance, $l_e/\xi_0 > 10$. The transition temperature of the four samples was (1.175 ± 0.001) K.

Experimental Procedure

The oscillator and aluminum sample were contained in a copper can which was immersed in the liquid-helium bath. The helium bath could enter the can through holes in the copper. A moly-permalloy magnetic shield surrounded the Dewars. A Stokes 6-in. booster pump produced temperatures down to 0.86 K. The temperature was measured with an Allen-Bradley 10-Ω 0.5-W carbon resistor with about half of the insulation removed. These resistors have the property that a plot of the logarithm of the resistance against the logarithm of the helium vapor pressure is a straight line.[10] Thus the resistors can be calibrated from the λ point down to about 1.2 K, and the resulting straight line can be extrapolated to give a temperature scale down to the lowest temperature. The relationship between bath pressure and resistance was checked using a McLeod gauge and was found to be sufficiently accurate.

The experimental procedure was first to calibrate the resistor and then to pump to the lowest temperature before turning the oscillator on. The bath could be slowly warmed up to the transition temperature of aluminum, while a plot of frequency versus bath pressure (and thermometer resistance) was made on the x-y recorder. In separate experiments, dc measurements of the transition temperature were made. With a short Cu wire in place of the sample, the change of the oscillator frequency over the temperature range of the experiment amounted to about 100 Hz at 14.8 MHz (see Fig. 4) and was reproducible to better than 5 Hz. Since a typical sample caused a frequency shift of about 2500 Hz over the same temperature range, the change in L can be known to better than 1%.

Because the oscillator could operate with the aluminum in the normal state, frequency measurements could be made for all temperatures up to T_c. However, the data reported are only for $t = T/T_c < 0.97$. At higher temperatures, df/dT became very large, and it was difficult to determine the temperature sufficiently accurately to plot the results in a reasonable way. It is also possible that dissipation effects were important in determining the resonant frequency in this temperature region.

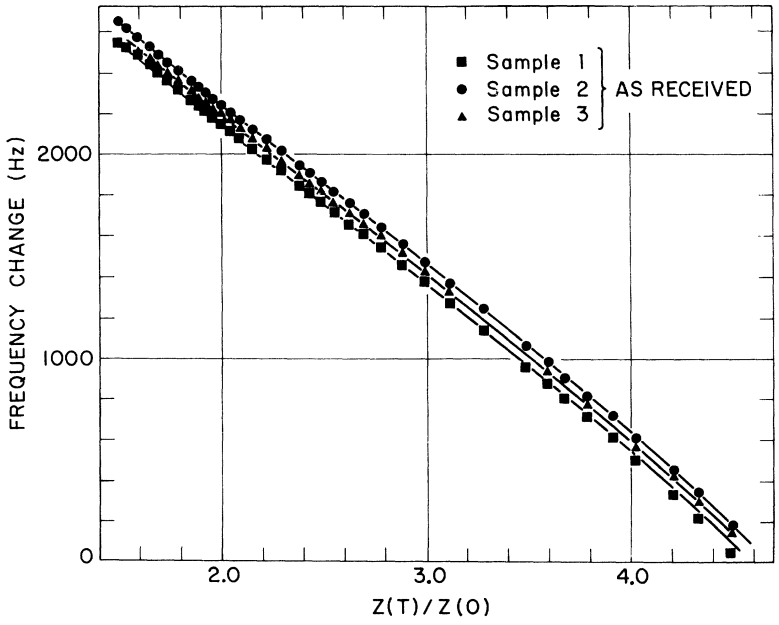

FIG. 5. Oscillator frequency change plotted as a function of the BCS local temperature dependence for three aluminum wire samples as received. The data have been displaced vertically for clarity.

The temperature dependence of the oscillator also was checked by inserting a sample of Pb wire which should show very little temperature dependence over the range of measurement. The results of this check agreed very well with the measurements done with the Cu wire.

RESULTS

Figure 5 shows the results of measuring the temperature dependence of the inductance of three aluminum wire samples in the condition in which they were received from the manufacturer. In this figure, the measured change in frequency is plotted as a function of Z, the temperature dependence of the penetration depth according to the local BCS theory. The experimental points have been displaced in a vertical direction to make comparison among the three measurements simpler. Such a displacement can be made since the absolute value of the frequency is not significant. The solid lines are simply the curve drawn through the experimental points of sample 2 displaced vertically to fit as well as possible the points of samples 1 and 3.

These samples are surprisingly similar in their temperature dependence over most of the temperature range. Above $Z = 3.8$ (corresponding to $t = 0.965$) there is a small, but noticeable, deviation of sample 1 from the other two, but over the rest of the range the three samples fit the same temperature dependence and have a very small scatter. This high reproducibility for samples measured on different days demonstrates the stability of the measuring equipment as well as the surprising uniformity of the wire as it was received.

Figure 5 also demonstrates that the temperature dependence of the penetration depth in the wire as received is not very different from that of the BCS local theory. The data very nearly fit a straight line for $Z \leq 3.6$ ($t \leq 0.96$). The slope of this line and Eq. (7) give for $d\lambda/dZ$ a value of 453 Å, a value that one might interpret as $\lambda(0)$ for the aluminum wire since scattering caused by lattice defects and surface roughness would tend to make the electrodynamics more nearly local in character.

Figure 6 shows the measured frequency change plotted as a function of $Z^*(t)/Z^*(0)$ for samples 1 and 2 as they were received, after annealing in a helium atmosphere for 6 h at a temperature of 600 °C, and finally after annealing and chemical polishing in a dilute solution of hydrofluoric acid. Sample 3 agreed well with samples 1 and 2 when measured as received and after annealing, but showed a somewhat higher slope than the other two after chemical polishing. It seems probable that

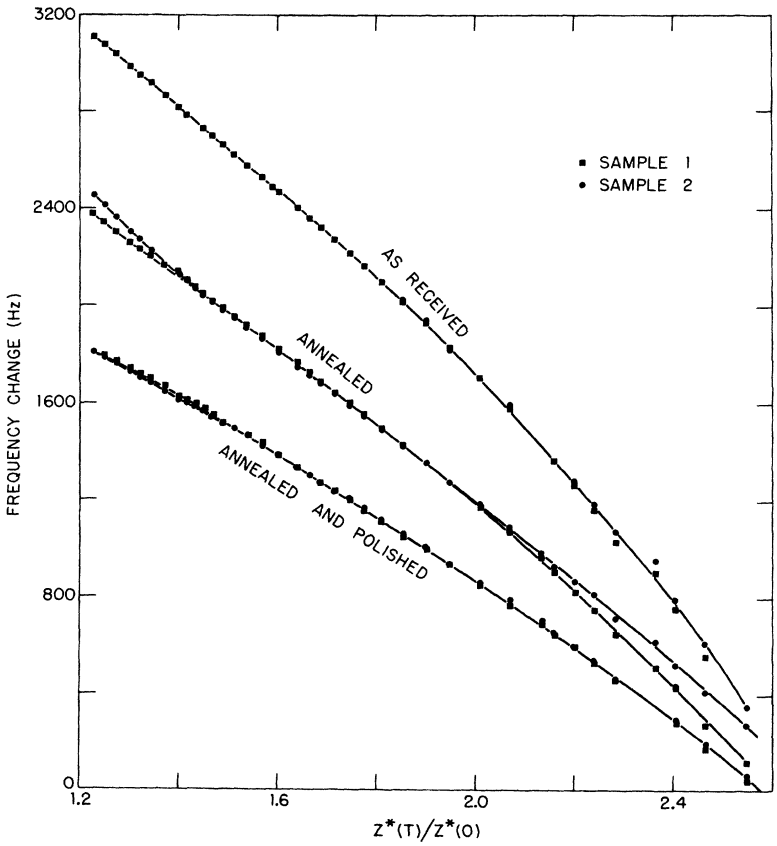

FIG. 6. Oscillator frequency change plotted as a function of the BCS nonlocal temperature dependence for samples 1 and 2 as they were received after annealing, and after annealing and polishing.

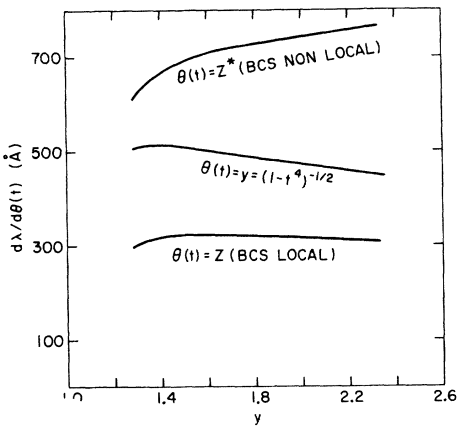

FIG. 7. $d\lambda/d\theta(t)$ as a function of y for the three temperature-dependent functions, $\theta = y(t)$, $\theta = Z(t)$, and $\theta = Z^*(t)$.

sample 3 was work hardened more than the other two from handling after it had been annealed. A fourth sample agreed with samples 1 and 2 at high temperatures, but in the lower temperature region there was a sudden break in the curve to a much higher slope. The cause of this anomalously high slope was not definitely determined, but the behavior was similar to that observed when the electrical contacts to the sample were faulty, and the results, assumed to be spurious, were discarded. All of the results that have been presented have been corrected for the temperature dependence of the oscillator circuit without a sample (Fig. 4).

The effect of impurities, of strain, of inhomogeneity, and of surface roughness is always to increase the penetration depth. The annealed and polished samples 1 and 2 probably represent the closest approach that we have been able to achieve to an ideal sample.

Assuming the temperature dependence of the penetration depth is given by Eq. (6), the first derivative $d\lambda/d\theta$ is a constant equal to $\lambda(0)$, the penetration depth at $T = 0$. Thus if we use the correct theoretical temperature dependence for $\theta(t)$, $d\lambda/d\theta$ (or what is proportional to it, $df/d\theta$) will be a constant. Figure 7 shows this derivative plotted for the average of the annealed and chemically polished samples 1 and 2 using for θ the three theoretical temperature dependences that were mentioned in the previous section, $y(t)$, $Z(t)$, $Z^*(t)$.

Values of df/dy were obtained by graphically measuring the slope of each of the two lowest curves in Fig. 6 at ten values of y. These were then converted to $d\lambda/dy$ by using Eq. (7) and the appropriate constants of the samples and measuring circuit shown in Table I. The middle curve of Fig. 7 shows the smoothed curve drawn through the average values of $d\lambda/dy$ for the two samples plotted against y. The other two curves were obtained by multiplying $d\lambda/dy$ by dy/dZ to obtain $d\lambda/dZ$ for the BCS local theory and by dy/dZ^* to obtain $d\lambda/dZ^*$ for the BCS extreme nonlocal theory.

If a given theoretical temperature dependence $\theta(t)$ were exactly correct, the result when plotted on Fig. 7 would be a straight horizontal line and the intercept at $y = 1$ ($t = 0$) would give the penetration depth $\lambda(0)$. The experimental results show that none of the three theoretical functions y, Z, or Z^* seems to fit exactly the measured result. However, the BCS local theory gives very nearly a constant $d\lambda/dZ$ and indeed is probably a constant within the limit of error since the two samples have their greatest divergence in the low-temperature region where $d\lambda/dZ$ turns down slightly. The function $d\lambda/dy$ is significantly not constant and the function $d\lambda/dZ^*$ varies considerably with temperature, especially in the low-temperature region where we would expect the extreme nonlocal theory to be most reliable. One obvious result is that the BCS extreme nonlocal theory does not describe the measured temperature dependence of the penetration depth of aluminum nearly as well as does the BCS local theory.

DISCUSSION

The value of $\lambda(0)$ for aluminum has been accepted as about 500 Å for over a decade. The original value given by Faber and Pippard[1] of 490 Å was based on the Gorter-Casimir temperature dependence $y(t)$. The BCS theory predicted[9] a value of 530 Å and the measurements of Biondi and Garfunkel[2] gave a value of 515 Å. Because of this apparent unanimity we originally attempted to explain the deviation of our results from other experiments by the nature of our samples, our treatment methods, or our analysis. However, when we actually plotted the results of the other measurements in the same way as our own, we found that they all agreed remarkably well as to the temperature dependence of the penetration depth of aluminum over the entire temperature range.

To exhibit this agreement with the four previous measurements, Fig. 8 shows $d\lambda/dy$ plotted as a function of y for all five of the measurements. In each case the measurements of previous experi-

TABLE I. Constants of samples and measuring circuit.

Sample	Diameter d (cm)	Length l (m)	Resonant circuit inductance L_0 (μH)	Resonant frequency f_0 (MHz)
1	0.0127	2.54	2.31	10.16
2	0.0127	2.46	2.30	10.18

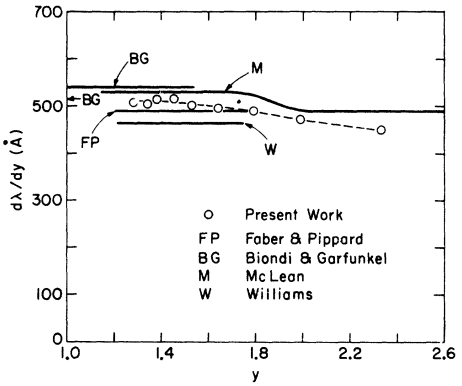

FIG. 8. Comparison of results of several measurements of $d\lambda/dy$ as a function of y.

ments have been consistent with the Gorter-Casimir dependence y. Thus the results are represented by solid horizontal lines with the exception of McLeans's[4] results which were given as two constant values above and below $y = 1.8$. The present measurements are represented by the circles and the dashed lines. The circles are actually the values obtained from the average of the slopes of the smoothed experimental curves. Their scatter gives some feeling for the repeatability of the present measurements. Apparently the present results are very close to the mean of the previous measurements and it is believed that there is little significance in their disagreement.

In Fig. 8, the empirical function y has been used because of tradition, making the graph more familiar than the theoretically more justified plots using the BCS function. However, if we assume that all the previous results agree within the limit of error with the present results when analyzed by the temperature dependence $y(t)$, they will also agree when analyzed by the BCS temperature dependences $Z(t)$ and $Z^*(t)$. Thus we may regard Fig. 7 as summarizing the data from all five experiments plotted according to the three different theoretical temperature dependences.

The most obvious conclusion is that $\lambda(t)$ does not have the BCS extreme nonlocal temperature dependence which we would expect for pure aluminum. The temperature dependence which fits the data best is the BCS local theory, with the empirical Gorter-Casimir dependence somewhere between the other two. If we take an average of the values of $\lambda(0)$ obtained over the temperature region used, the value for the local BCS theory is 315 Å, for the empirical theory is 490 Å, and for the BCS extreme nonlocal theory is 725 Å. However, it should be mentioned that only in the case of the BCS local theory is there any reason to believe that this value of the derivative $d\lambda/dZ$ is the true penetration depth $\lambda(0)$, since only in this case is the derivative a constant. This resolution of the difficulty is quite unsatisfactory though, because the BCS theory predicts that aluminum is an extreme nonlocal superconductor, and without the nonlocal correction, it predicts the value of $\lambda(0) = 157$ Å which is not close to that measured.

The inconsistencies described above allow no unambiguous conclusion concerning the penetration depth of aluminum. We therefore examine some of the assumptions made in the theory and the interpretation of the experiment to see which might be unjustified.

In the theory, the most vulnerable assumption is perhaps that the temperature dependence of the energy gap, $\Delta(t)/\Delta(0)$, is accurately given for aluminum by the BCS theory. The function $\Delta(t)/\Delta(0)$ from which $\lambda(t)/\lambda(0)$ follows is calculated with several simplifying assumptions in which the detailed metallic properties of aluminum are ignored. To investigate the effect of these simplifications, we could use empirically measured values of $\Delta(t)/\Delta(0)$ in place of the theoretical function. The original tunneling measurements by Giaever and Megerle[11] have too much scatter to give a useful temperature dependence for the present purpose. Douglass and Meservey,[12] for a range of thickness from 400 to 10 000 Å, obtained at lower temperatures a smaller increase in the normalized gap than predicted by the BCS theory. In this case the criterion used to determine the gap has been criticized, but in the low-temperature region, change of the criterion makes little difference in $\Delta(t)/\Delta(0)$. The deviation observed would change the extreme nonlocal curve in Fig. 7 to one which deviates somewhat less from a horizontal line. On the other hand, Cherney and Shewchun[13] have observed in extremely thin films (35–80 Å) the exact BCS temperature dependence. However, even if we were certain of these measured values, the application to bulk aluminum of $\Delta(t)/\Delta(0)$ obtained from thin films is of rather dubious validity. The only measurements of the temperature dependence of the energy gap in pure bulk aluminum by Biondi and Garfunkel[2] agree with the BCS temperature dependence, but the uncertainty of measurement is large enough to give us only limited help in the present dilemma. We can conclude that, although a small deviation from the BCS temperature dependence of the energy gap may exist in aluminum which would change the curve $d\lambda/dZ^*$ somewhat, this effect probably could not account for all of the departure of $d\lambda/dZ^*$ from a constant.

CONCLUSIONS

The conclusions we reach from the data and its analysis are the following.

(i) The experimental results of the present low-frequency measurements agree with previous high-frequency measurements on the temperature depen-

dence of the superconducting surface inductance and of $\lambda(t)/\lambda(0)$ of pure aluminum.

(ii) This temperature dependence of λ does not agree with the BCS extreme nonlocal theory using the theoretical value of $\Delta(t)/\Delta(0)$.

(iii) The agreement with the BCS theory can perhaps be improved by using an empirically measured function for $\Delta(t)/\Delta(0)$, but the absolute value calculation of $\lambda(0)$ would still not agree with experiment.

(iv) Because of these uncertainties in the theoretical temperature dependence, the problem will be advanced only by a good measurement of the absolute value of the penetration depth.

ACKNOWLEDGMENTS

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APPENDIX

If a time-dependent magnetic field is applied parallel to the surface of a semi-infinite superconductor starting at time $t=0$, the total energy stored in the superconductor is

$$U=\int_0^t(\vec{E}_0\times\vec{H}_0)\,dt\ , \tag{16}$$

where E_0 and H_0 are the parallel components of the electric and magnetic fields at the surface. E_0 can be given in terms of the penetration depth λ using Faraday's law

$$\oint\vec{E}\cdot d\vec{l}=-\int\frac{\partial\vec{B}}{\partial t}\,ds. \tag{17}$$

If the contour of integration extends well inside the superconductor so that $E=0$ and $B=0$, we obtain

$$E_0=-\frac{\partial\Phi}{\partial t}=-\frac{\partial}{\partial t}(\mu_0\lambda H_0)\ , \tag{18}$$

where Φ is the magnetic flux.

The last equality comes from the definition of the penetration depth λ:

$$\lambda\equiv\frac{1}{\mu_0 H_0}\int_0^\infty B\,dx=\frac{\Phi}{\mu_0 H_0}\ . \tag{19}$$

From Eqs. (16) and (18) the total stored energy is

$$U=\int_0^t\left(\mu_0\lambda H_0\frac{\partial H_0}{\partial t}\right)dt=\tfrac{1}{2}\mu_0\lambda H_0^2\ . \tag{20}$$

For all but the highest frequencies the displacement current may be neglected and I, the surface current per unit width, equals H_0. The energy per unit area is, then,

$$U=\tfrac{1}{2}\mu_0\lambda I^2\ . \tag{21}$$

The energy definition of total inductance,

$$U=\tfrac{1}{2}LI^2\ , \tag{22}$$

allows us to identify the total inductance per unit surface area as

$$L=\mu_0\lambda\ .$$

This result is the same as that obtained by applying the rule that $L=\Phi/I$. The result depends in no way on whether the electrodynamics are local or nonlocal within the superconductor, since no relation between $\vec{J}$ and $\vec{A}$ or $\vec{J}$ and $\vec{E}$ inside the superconductor was used in its derivation. The essential condition for this simple result is that the thickness or diameter $d\gg\lambda$, so that $\vec{E}$ and $\vec{B}$ vanish inside the superconductor. When this condition is not met, as for a very thin wire or a thin-film cylinder, the present method of applying a given current along the sample would give a measured inductance different from that obtained by the usual method of imposing a known magnetic field parallel to the length of the cylindrical sample. In the limit $d\ll\lambda$ the present method would measure the kinetic inductance which would increase without limit as $d/\lambda\to 0$. On the other hand, the method of applying a longitudinal magnetic field would measure an inductance difference between the normal and superconducting state which tends to zero as $d/\lambda\to 0$. However, since $d\gg\lambda$ in the present measurements, the result of Eq. (2) is valid.

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[1]T. E. Faber and A. B. Pippard, Proc. Roy. Soc. (London) A231, 336 (1955).

[2]M. A. Biondi and M. P. Garfunkel, Phys. Rev. 116, 862 (1959).

[3]D. L. Williams, Proc. Phys. Soc. (London) 79, 594 (1962).

[4]W. L. McLean, Proc. Phys. Soc. (London) 79, 572 (1962).

[5]R. Meservey and P. M. Tedrow, J. Appl. Phys. 40, 2028 (1969).

[6]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[7]B. Mühlschlegel, Z. Physik 155, 313 (1959).

[8]A. B. Pippard, Proc. Roy. Soc. (London) A216, 547 (1953).

[9]J. Bardeen and J. R. Schrieffer, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland, Amsterdam, 1961), Vol. 3, p. 170.

[10]S. Cunsolo, M. Santini, and M. Vicenti-Missoni, Cryogenics 5, 168 (1965).

[11]I. Giaever and K. Megerle, Phys. Rev. 122, 1101 (1961).
[12]D. H. Douglass, Jr. and R. Meservey, Phys. Rev. 135, A19 (1964).
[13]O. A. E. Cherney and J. Shewchun, Can. J. Phys. 47, 1101 (1969).

PHYSICAL REVIEW B VOLUME 4, NUMBER 1 1 JULY 1971

Pressure Dependence of the Specific Heat of a Superconductor Derived from Critical-Magnetic-Field Data*

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The low-temperature variation of the critical magnetic field of a superconductor is related to the normal-state electronic specific-heat coefficient γ. Measurements of the critical magnetic field at zero and finite pressure are used to obtain a value for the volume Ω dependence of γ. We show that strong-coupling corrections must be included in such an analysis. For example, in Pb, ignoring such corrections leads to an error of almost a factor of 2 in $d\ln\gamma/d\ln\Omega$.

The temperature variation of the critical magnetic field $H_c(T)$ of strong-coupling superconductors, like Pb and Hg, is known to deviate from the predicted BCS behavior.[1,2] It is customary to introduce a function $D(t)$ defined by

$$D(t) = H_c(T)/H_c(0) - [1 - (T/T_c)^2] \ , \qquad (1)$$

where t is the reduced temperature T/T_c and T_c is the critical temperature. For Pb, $D(t)$ is found to be positive, in contrast to Al for which $D(t)$ is negative in close agreement with BCS theory.

When the electron-phonon interaction becomes particularly large, the details of the interactions involved become of some importance and the Eliashberg[3] formulation of pairing theory must be used instead of the simple BCS model.[4] On the basis of the Eliashberg gap equations, it is possible to understand quantitatively[5] not only the observed variation of $D(t)$ with t but also many other anomalous properties of superconducting Pb.[6-8] For instance, the very large value observed for the ratio

$$2\Delta_0/K_B T_c$$

(where Δ_0 is the energy gap at zero temperature, and K_B is the Boltzmann constant) is explained.

On application of hydrostatic pressure P, the critical temperature of Pb is observed to decrease. The ratio $2\Delta_0/K_B T_c$ is also found to decrease[9] and tends toward the weak-coupling BCS value of 3.53. These effects are understood[10] theoretically and are a simple consequence of a reduction in the electron-phonon interaction as P increases. Thus, we would also expect that the shape of $D(t)$ vs t would change under pressure and tends toward the BCS temperature variation. This would mean a breakdown of the "similarity principle" which is sometimes introduced in discussions of the pressure dependence of $H_c(T)$.[11,12] According to this principle the shape of $D(t)$ would be invariant. It is the purpose of this paper to make an estimate of the volume dependence of the shape of $D(t)$ vs t and to explore the consequences of such an estimate. We will limit the discussion to very low temperature (i.e., $t \to 0$) and consider only Pb.

Denote the slope of the low-temperature behavior of $D(t)$ vs t^2 by α. For small t we have

$$D(t) \cong \alpha t^2 .$$

The following thermodynamic relationship involving α is easily obtained[8]:

$$\gamma = [H_c^2(0)/2\pi T_c^2](1 - \alpha) , \qquad (2)$$

where γ is the coefficient of the electronic specific heat in the normal state at low temperature. The relation (2) is often used to obtain information on the volume dependence of the specific heat from the critical-field data.[11,12] We will return to this important point later. For the moment we proceed with an estimate of the volume dependence of α.

Recently Carbotte and Vashishta[13] have calculated the zero-temperature condensation energy U of a number of superconductors. Their results are obtained from detailed solutions of the Eliashberg equations based on realistic values for the interaction kernels. The kernels can be obtained accurately by "inversion" of superconducting tunneling data.[7] The value of U obtained in this way is expected to be quite reliable. It is convenient to write the results for U as